Journal of Statistical Physics, Vol. 30, No. 2, 1983

Fractal and Lacunary Stochastic Processes

B. D. Hughes,[1] E. W. Montroll,[2] and M. F. Shlesinger[2,3]

Discrete-time random walks simulate diffusion if the single-step probability density function (jump distribution) generating the walk is sufficiently short-ranged. In contrast, walks with long-ranged jump distributions considered in this paper simulate Lévy or stable processes. A one-dimensional walk with a self-similar jump distribution (the Weierstrass random walk) and its higher-dimensional generalizations generate fractal trajectories if certain transience criteria are met and lead to simple analogs of deep results on the Hausdorff–Besicovitch dimension of stable processes. The Weierstrass random walk is lacunary (has gaps in the set of allowed steps) and its characteristic function is Weierstrass' non-differentiable function. Other lacunary random walks with characteristic functions related to Riemann's zeta function and certain number-theoretic functions have very interesting analytic structure.

KEY WORDS: Random walks; fractals; stable distributions; lacunary series.

1. INTRODUCTION

A persistent theme of statistical physics, recurring in a wide variety of distinct models,[1] is the insensitivity of the properties of the bulk (i.e., a large, or mathematically infinite system) to the precise microstructure of its constituent parts. This insensitivity to microstructure is not unconditional, and requires (i) restrictions on the spatial dimension of the system and (ii) certain moment conditions upon statistical distributions and ranges of interactive forces.

Presented at the Symposium on Random Walks, Gaithersburg, MD, June 1982.
[1] Department of Chemical Engineering and Materials Science, University of Minnesota, Minneapolis, Minnesota 55455. Supported by the U.S. Department of Energy.
[2] Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.
[3] La Jolla Institute, La Jolla, California 92038. Supported by a grant from DARPA.

0022-4715/83/0200-0273$03.00/0

Perhaps the simplest problem of statistical physics in which these matters are well documented is the discrete-time random walk, where the question of microstructure sensitivity is settled by the central limit theorem and its extensions.[2] If $p(\mathbf{x})$ is the single-step probability density function ("jump distribution") then qualitative behavior insensitive to the precise structure of $p(\mathbf{x})$ [embodied in a diffusive, or Gaussian, continuum limit] is guaranteed provided that the mean-squared displacement per step

$$\langle \mathbf{x}^2 \rangle = \int \mathbf{x}^2 p(\mathbf{x})\, d\mathbf{x} \tag{1}$$

is finite. When this constraint is violated weakly, i.e., if the integral defining $\langle \mathbf{x}^2 \rangle$ diverges only mildly, a diffusive continuum limit can still be exhibited, but only at the cost of introducing a nonstandard scaling of length and time to define the diffusion constant. When the integral diverges sufficiently strongly, the continuum limit is a Lévy (stable) distribution,[3] the order of which depends on the asymptotic behavior of $p(\mathbf{x})$ as $|\mathbf{x}| \to \infty$. The probability density function $P(\mathbf{x}, t)$ for the position $\mathbf{x}$ at time t of a particle whose motion is governed by a stable distribution is most easily characterized in Fourier space. For the symmetric, one-dimensional case

$$\tilde{P}(q,t) \equiv \int_{-\infty}^{\infty} e^{iqx} P(x,t)\, dx = \exp[-Ct|q|^{\mu}] \tag{2}$$

where C is a positive constant and the order μ of the distribution is restricted to the interval $0 < \mu \leqslant 2$. The case $\mu = 2$ corresponds to the Gauss distribution and is the only case for which the mean-squared displacement at any instant is finite.

An example of Gillis and Weiss[4] can be used for illustrative purposes. Consider a one-dimensional lattice walk for which the probability $p(l)$ for a displacement of l sites at a given step is

$$p(l) = \frac{1}{2\zeta(1+\alpha)} \sum_{n=1}^{\infty} n^{-1-\alpha} \{\delta_{l,n} + \delta_{l,-n}\} \tag{3}$$

where $\alpha > 0$ and ζ denotes the Riemann zeta function,[5] defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{4}$$

When $\alpha > 2$, the walk has a diffusive continuum limit, while if $\alpha < 2$ the continuum limit is given by the stable distribution (2) of order α. This change in the statistical properties of the walk can be traced to a qualitative change in the structure function

$$\lambda(k) = \sum_{l=-\infty}^{\infty} e^{ikl} p(l) \tag{5}$$

of the walk as α passes through the value 2. For $\alpha > 2$,

$$\lambda(k) - 1 \sim -\frac{1}{2}\left\{\sum_{l=-\infty}^{\infty} l^2 p(l)\right\}k^2 \quad \text{as} \quad k \to 0 \tag{6}$$

while if $0 < \alpha < 2$

$$\lambda(k) - 1 \sim -\text{const} \cdot |k|^{\alpha} \quad \text{as} \quad k \to 0 \tag{7}$$

Borrowing the terminology of statistical thermodynamics, we have a change in the "critical exponent" $\mu \equiv \lim_{|k| \to 0}\{\ln[1-\lambda(k)]/\ln|k|\}$ since $\mu = 2$ for $\alpha > 2$ and $\mu = \alpha$ for $0 < \alpha < 2$.

In a given spatial dimension s, all symmetric lattice walks with finite mean-squared displacement per step have, in addition to a common Gaussian limit, qualitatively similar random walk statistics, of which three preeminent examples are

(i) R, the probability of eventual return to the origin[6]:

$$R = 1 \quad \text{if} \quad s = 1 \text{ or } 2; \qquad R < 1 \quad \text{if} \quad s \geqslant 3$$

(ii) τ_c, the conditional mean time for any walker who ever returns to the origin to make his first return[7]:

$$\tau_c = \infty \quad \text{if} \quad s \leqslant 4; \qquad \tau_c < \infty \quad \text{if} \quad s \geqslant 5$$

(iii) S_n, the mean number of distinct sites visited in a walk of n steps,[8] where $n \to \infty$:

$$S_n \propto \sqrt{n} \quad \text{if} \quad s = 1; \qquad S_n \propto n/\ln n \quad \text{if} \quad s = 2; \qquad S_n \propto n \quad \text{if} \quad s \geqslant 3$$

Each of these random walk statistics exhibits a "critical dimensionality" above which the statistic is qualitatively independent of s. If the condition of finiteness of $\langle \mathbf{x}^2 \rangle$ is relaxed, the critical dimensionalities are shifted. For example in the Gillis–Weiss walk in the case $\alpha = 1$, $S_n \propto n/\ln n$, while if $\alpha < 1$, $S_n \propto n$.

2. FRACTAL RANDOM WALKS

It has become apparent in recent years, principally from the work of Mandelbrot[9] that fractional exponents, shifts in effective dimensionality and microstructure-dependent behavior may be linked to self-similar scaling and microstructure in space and time. However, this link is not transparent for simple discrete-time random walks, such as the example of Gillis and Weiss, or for their continuum analogs, the stable processes (Lévy flights). To exhibit the link more clearly, the authors considered in earlier papers[10,11] a class of random walks with a self-similar microstructure built into the jump distribution. The prototype for this class of walks takes place on the one-dimensional continuum, with jump distribution

$$p(x) = \frac{a-1}{2a} \sum_{n=0}^{\infty} a^{-n} \{\delta(x - \Delta b^n) + \delta(x + \Delta b^n)\} \tag{8}$$

Here a and b are dimensionless and greater than 1, while Δ is a length scale introduced to allow passage to a continuum limit ($\Delta \to 0$ and $\tau \to 0$ jointly, where τ is the time between steps, the relative rates of decrease of Δ and τ being appropriately matched). When b is integral, the walk is confined to a lattice of spacing Δ, and its structure function is

$$\lambda(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n} \cos(b^n k) \tag{9}$$

More generally, when b is not restricted to integer values, the characteristic function of (8) is

$$\tilde{p}(q) = \int_{-\infty}^{\infty} e^{iqx} p(x)\, dx = \lambda(q\Delta) \tag{10}$$

The random walk generated by (8) can be connected to Mandelbrot's fractals in three different ways. In each of these connections, which we now describe, the quantity

$$\mu = \ln a / \ln b \tag{11}$$

acquires a special significance. We focus our attention on the parameter range $0 < \mu \leqslant 2$ ($b^2 \geqslant a$) where

$$\langle x^2 \rangle = \frac{a-1}{a} \Delta^2 \sum_{n=0}^{\infty} [b^2/a]^n \tag{12}$$

is infinite.

2.1. Intuitive Connection

The jump distribution (8) ensures that a step of length Δb^n is a times more likely than the next longest step (Δb^{n+1}) and, on the average, about a jumps of a given length Δb^n will be made for each jump of length Δb^{n+1}. As the walk progresses, the set of sites visited should consist of clusters of sites, punctuated by gaps, which are part of a hierarchy of clusters which is self-similar in some statistical sense. Since the parameters a and b are measures of the number of subclusters in a cluster and the spatial size scaling between clusters, the parameter μ in (11) may be identified as the analog of a fractal dimension in the sense of Mandelbrot.[9] However, an important restriction must be imposed. As the random walk evolves, two outcomes are possible—the walker may visit all sites (persistence) or fail to visit some sites (transience). (This dichotomy is strictly true as stated only for lattice walks. For walks on the continuum, one speaks not of visiting a point, but rather of visiting an arbitrarily small neighborhood of the point.) For the present model, transience is certain when $0 < \mu < 1$, and persistence is certain when $\mu \geqslant 1$. In the persistent case, the walker ultimately returns to fill in any gaps in the set of sites visited and the clustering eventually disappears.

2.2. Connection via a Continuum Limit

It can be shown[10,11] that if Δ and τ (the characteristic length scale, and the time between steps) approach zero jointly, while

$$a \sim 1 + \alpha\Delta \quad \text{and} \quad b \sim 1 + \beta\Delta \tag{13}$$

subject to the constraints that $\mu = \ln a / \ln b$ and Δ^{μ}/τ are held constant, then the probability density function $P(x, t)$ for the position x of the walker after a time t is given by the one-dimensional symmetric stable distribution of order μ [Eq. (2)]. The random walk thus simulates stable processes, which figure prominently in the theory of fractals. Although the walk has two parameters $\underline{a}$ and b we observe that they do not enter individually, but only in the form of a fractal dimension $\ln a/\ln b$.

2.3. Connection via a Scaling Equation

The characteristic function for the walk, defined by (8) and (9), obeys a scaling equation, since

$$\lambda(k) = a^{-1}\lambda(bk) + \frac{a-1}{a}\cos(k) \tag{14}$$

The nonanalytic part $\lambda_{\text{sing}}(k)$ of $\lambda(k)$ satisfies the homogeneous equation

$$\lambda_{\text{sing}}(k) = a^{-1}\lambda_{\text{sing}}(bk) \tag{15}$$

so that

$$\lambda_{\text{sing}}(k) = |k|^{\mu} Q(k) \tag{16}$$

where $Q(k)$ is a bounded function, periodic in $\ln|k|$ with period $\ln b$ of quite intricate structure.[10,11] The parameter μ thus enters as a nonintegral critical exponent derived from a scaling equation (so that the random walk problem has some similarities to real-space renormalization group transformations, which have been explored elsewhere[12]).

By exploiting simultaneously the intuitive connection to fractals and the connection to Lévy's stable processes, we are able to derive simple analogs for our random walk of deep theorems on the Hausdorff–Besicovitch dimension of stable processes. It may be helpful to the reader to note here that the *Hausdorff–Besicovitch dimension* D_H of a subset Ω of E-dimensional Euclidean space is, for self-similar sets, the *fractal dimension* Mandelbrot[9] introduces by scaling arguments. However, a precise definition of D_H can be given,[13] which applies to arbitrary sets of points Ω, and D_H can be shown to satisfy the inequality

$$D_T \leqslant D_H \leqslant E \tag{17}$$

where D_T (an integer) is the *topological dimension*[14] of the set Ω. Mandelbrot calls the set Ω a *fractal* if $D_H > D_T$.

McKean[15] has shown that for a one-dimensional stable process of order μ, with $x(t, \omega)$ denoting the position after time t in any particular realization ω of the process,

$$P\{x(t, \omega) = x_0 \text{ for some } t > 0\} = 0 \tag{18}$$

for every real x_0, provided that $0 < \mu < 1$. In other words, a stable process of order $\mu < 1$ skips over any point x_0 almost surely. (Here P denotes probability measure over the space of all realizations ω of the process, and an event happens almost surely if it occurs for all realizations ω except for a set of measure zero.) McKean has also shown that the range of $x(t, \omega)$, i.e., the set of points visited, almost surely has Hausdorff–Besicovitch dimension μ if $0 < \mu < 1$, and 1 if $\mu \geqslant 1$.

Our random walk is transient if $\mu < 1$ and persistent if $\mu \geqslant 1$, so that we have an immediate analog of McKean's first theorem. When the walk is transient, μ is the analog of the fractal dimension of the set of sites visited, while if the walk is persistent we fill up the entire space available to the walker; this is the analog of McKean's second theorem.

The one-dimensional random walk we have considered is merely a prototype for a class of random walk and renewal processes[10,11] which one may call *fractal stochastic process*. We give here another example from this class[11]—a random walk on a s-dimensional continuum generated by the jump distribution

$$p(\mathbf{x}) = \left[A_s|\mathbf{x}|^{s-1}\right]^{-1} p_1(|\mathbf{x}|) \tag{19}$$

where A_s is the surface area of the unit sphere in s-dimensions and $p_1(|\mathbf{x}|)$ is the probability density function for the length of any step, given by

$$p_1(|\mathbf{x}|) = \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n} \delta(|\mathbf{x}| - \Delta b^n) \tag{20}$$

The characteristic function for the distribution (19) can be shown[11] to be

$$\begin{aligned} \tilde{p}(\mathbf{q}) &= \int \exp(i\mathbf{q} \cdot \mathbf{x}) p(\mathbf{x})\, d^s\mathbf{x} \\ &= \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n} \Gamma(\tfrac{1}{2}s)(\tfrac{1}{2}|\mathbf{q}|\Delta b^n)^{1-s/2} J_{s/2-1}(|\mathbf{q}|\Delta b^n) \end{aligned} \tag{21}$$

with J_ν denoting the usual Bessel function of the first kind of order ν. This "fractal Rayleigh–Pearson walk" has the same three connections to fractals as the simpler one-dimensional example. It is transient when $\mu < 2$, so that we have a simple analog of the theorem of Blumenthal and Getoor[16] which states that the Hausdorff–Besicovitch dimension of the range of an s-dimensional isotropic stable process ($s \geqslant 2$) of order μ is almost surely μ.

3. LACUNARY RANDOM WALKS

The "clustered" random walk in one dimension discussed in the preceding section may be called a *fractal random walk*, in view of its many connections to fractals. It may also be described as a *lacunary random walk* [with "lacuna" meaning "hole" or "gap"], since there are gaps in the set of possible steps, and the characteristic function can be related to lacunary Taylor[17] and Fourier[18] series. If b is an integer, defining $z = e^{ik}$ we can rewrite the definition (9) of the structure function of the walk in the form

$$\lambda(k) = \frac{a-1}{a} \operatorname{Re}\left\{ \sum_{n=0}^{\infty} a^{-n} z^{b^n} \right\} \tag{22}$$

The power series has gaps, or missing terms. For example, if $b = 2$ we have

$$1 + \text{gap of } 1 + a^{-1}z^2 + \text{gap of } 1 + a^{-2}z^4$$
$$+ \text{gap of } 3 + a^{-3}z^8 + \text{gap of } 7 + a^{-4}z^{16} + \cdots$$

The gaps increase in size. It can be shown[17] that for any $\underline{a} > 1$ and integer $b > 1$, the power series in (22) converges for $|z| \leqslant 1$, but it cannot be analytically continued beyond the unit circle due to a dense crowding of nonanalytic points on the unit circle. Noncontinuability is a characteristic property of lacunary power series [provided that the gaps grow sufficiently rapidly] exemplified in the remarkable theorem of Fabry[19]: *If* $\lambda_n/n \to \infty$ *and the radius of convergence of* $f(z) = \sum a_n z^{\lambda_n}$ *is* 1, *then* $f(z)$ *cannot be continued beyond* $|z| = 1$.

The noncontinuability of the series in (22) suggests that the structure function $\lambda(k)$ has extremely complicated behavior as a function of k. In fact, for $\mu \leqslant 1$, $\lambda(k)$ is Weierstrass' example of a function which at no point possesses a finite derivative,[20] and the authors have taken the liberty of calling the random walk of which it is the structure function (or characteristic function) *Weierstrass' random walk*.[10]

It is interesting to consider other random walks for which the structure function is a lacunary series. Perhaps the simplest example is the one-dimensional lattice walk generated by

$$p(x) = \{2\zeta(1+\alpha)\}^{-1} \sum_{n=1}^{\infty} n^{-1-\alpha} \{\delta(x - \Delta n^{\beta}) + \delta(x + \Delta n^{\beta})\} \tag{23}$$

where $\alpha > 0$, β is real and greater than 1 and ζ denotes Riemann's zeta function [Eq. (4)]. Since

$$\langle x^2 \rangle = \{\Delta^2/\zeta(1+\alpha)\} \sum_{n=0}^{\infty} n^{2\beta-\alpha-1}$$

$$= \begin{cases} \Delta^2\zeta(\alpha+1-2\beta)/\zeta(\alpha+1) & \text{if } \alpha > \beta \\ \infty & \text{if } \alpha \leqslant 2\beta \end{cases} \tag{24}$$

the walk is nondiffusive when $\alpha < 2\beta$; we restrict our attention to this case. For $\beta > 1$, the walk is *lacunary*, while if $\beta = 1$ it is the example of Gillis and Weiss.[4] For integral $\beta > 1$, the structure function of the walk is

$$\lambda(k) = \zeta(1+\alpha)^{-1} \sum_{n=1}^{\infty} n^{-1-\alpha}\cos(n^{\beta}k) \tag{25}$$

which is both a generalization of Riemann's zeta function and a lacunary series. The case $\beta = 2$ has been analyzed in detail by Hardy,[20] together with the analogous series with the cosine replaced by a sine, in the first rigorous investigation of Riemann's celebrated assertion that the function $\sum_{n=1}^{\infty} n^{-2}\sin(n^2x)$ is nondifferentiable.[21] Hardy proved that if $\beta = 2$ and $\alpha < 3/2$, the function $\lambda(k)$ defined by (25) and the analogous sine series do not have a finite derivative with respect to k at any point which is an irrational multiple of π. However, at some rational multiples of π a finite derivative can be shown to exist.

We call the random walk generated by (23) the *Riemann walk*. It can be shown rigorously by Mellin transform techniques (similar to those used in Refs. 10–12) that if

$$\nu \equiv \alpha/\beta > 1/2 \tag{26}$$

then

$$\lambda(k) - 1 \sim -\frac{\pi|k|^{\nu}}{2\beta\zeta(1+\alpha)\Gamma(1+\nu)\sin(\frac{1}{2}\pi\nu)} \tag{27}$$

as $k \to 0$. If, in addition to the condition that $\nu > 1/2$, we restrict β to the interval $0 < \beta \leqslant 2$, the following asymptotic expansion holds:

$$\lambda(k) \sim 1 - \frac{\pi|k|^{\nu}}{2\beta\zeta(1+\alpha)\Gamma(1+\nu)\sin(\frac{1}{2}\pi\nu)} + \sum_{n=1}^{\infty} \frac{(-1)^n\zeta(1+\alpha-2n\beta)}{(2n)!\,\zeta(1+\alpha)} k^{2n} \tag{28}$$

The series on the right-hand side is convergent for all k when $0 < \beta < 1$, and divergent for all $k > 0$ when $\beta > 1$. For $\beta = 1$ (the Gillis–Weiss lattice walk), the series converges only for $|k| \leqslant 2\pi$, but the value of $\lambda(k)$ at other values of k can be inferred from the periodicity of $\lambda(k)$. Although the Riemann walk is fractal, in the sense that its continuum limit is a Lévy distribution, it does not possess the precise self-similar scaling manifested in the Weierstrass random walk.

Other lacunary random walks can be constructed using functions of great importance in analytic number theory. Consider the two lacunary lattice walks having structure functions

$$\lambda(k) = C \sum_{\substack{n=2 \\ n \text{ prime}}}^{\infty} n^{-a}\cos(nk) \tag{29}$$

and

$$\lambda(k) = K \sum_{n=1}^{\infty} n^{-a}|\mu(n)|\cos(nk) \tag{30}$$

where C and K are constants chosen to ensure normalization $[\lambda(0) = 1]$ and $\mu(n)$ denotes the Möbius function[5]

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if any prime factor of } n \text{ is repeated} \end{cases} \tag{31}$$

Let $\pi(x)$ denote the number of prime numbers not exceeding x, so that

$$\pi(n+1) - \pi(n) = \begin{Bmatrix} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{Bmatrix} \tag{32}$$

Then in view of the identities[5]

$$\sum_{\substack{n=2 \\ n \text{ prime}}}^{\infty} n^{-p} = p\int_2^{\infty} \pi(x)x^{-p-1}\,dx, \qquad \sum_{n=1}^{\infty} n^{-p}|\mu(n)| = \zeta(p)/\zeta(2p) \tag{33}$$

and the contour integral representation

$$\cos(k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} |k|^{-s}\Gamma(s)\cos\left(\frac{\pi}{2}s\right) ds, \qquad 0 < c = \mathrm{Re}(s) < 1 \tag{34}$$

we can show that

$$\lambda(k) - 1 = \frac{C}{2\pi i}\int_{c-i\infty}^{c+i\infty} |k|^{-2}\Gamma(s)\cos\left(\frac{\pi}{2}s\right)(s+a)\int_2^{\infty} \pi(x)x^{-s-a-1}\,dx\,ds \tag{35}$$

and

$$\lambda(k) - 1 = \frac{K}{2\pi i}\int_{c-i\infty}^{c+i\infty} |k|^{-s}\Gamma(s)\cos\left(\frac{\pi}{2}s\right)\frac{\zeta(s+a)}{\zeta(2s+2a)}\,ds \tag{36}$$

for the prime and Möbius walks, respectively, with the value of c suitably restricted. The leading order small k behavior of the right-hand side of (35) comes from the singularity of the integrand at $s = 1 - a$. This singularity is not a simple pole, in view of the prime number theorem[5] which, in its weakest form, states that $\pi(x) \sim x/\ln x$ as $x \to \infty$. It would be of interest if (35) could be used as the basis of a new proof of the prime number theorem. Equation (36) for the Möbius walk raises more interesting possibilities. While the dominant behavior comes from the simple pole at $s = 1 - a$, the higher-order behavior is heavily influenced by the zeros of $\zeta(2s + 2a)$. It is well known that apart from the trivial zeros at $s = -2, -4, -6 \ldots$ the function $\zeta(s)$ has zeros only in the *critical strip* $0 < \mathrm{Re}(s) < 1$, and the famous *Riemann hypothesis* asserts that all the zeros lie on the critical line $\mathrm{Re}(s) = 1/2$. It has been stated[22] that the proof or disproof of the Riemann hypothesis is "unquestionably the most celebrated question in mathematics." Perhaps the connection between it and random walks, as exhibited here, might provide some clue to the true nature of the difficulties impeding resolution of this problem. Both random walks of Eqs. (35) and (36) can be made to converge to the same Lévy flight in the continuum limit. It is the rate of convergence that is different and determined by the higher-order terms in the structure function.

Our discussion of lacunary random walks has been largely confined here to one-dimensional cases, but generalizations to higher-dimensional walks are possible. For example, the characteristic function of the fractal Rayleigh–Pearson walk[11] [Eq. (21)] is a kind of lacunary series: since $\sum c_n J_0(nx)$ is called a Schlömilch series, it may be described by a lacunary generalized Schlömilch series. Although such series are sure to possess many beautiful properties analogous to those of lacunary Taylor and Fourier series, they appear to have never been studied.

Although our discussion of fractal and lacunary stochastic processes has centered here on discrete time random walks, the basic ideas can be extended to other processes, including renewal processes[12] and continuous-time random walks. In the latter case, the probability density function $P(\mathbf{x}, t)$ for the position of the walker at time t then satisfies a linear generalized master equation.[24] Such equations can also be derived from nonlinear deterministic equations (either through a Carleman linearization technique[25] or via a method of Nishigori[26] which gives the memory kernel in terms of a continued fraction). The connection between "chaos" in nonlinear deterministic systems and spatial or temporal fractal behavior in random walk systems will be investigated elsewhere.

REFERENCES

1. (a) C. J. Thompson, *Contemp. Phys.* **19**:203 (1978); (b) M. Sahimi, B. D. Hughes, L. E. Scriven, and H. T. Davis, *J. Chem. Phys.*, to appear; (c) A. Ben-Mizrahi and D. J. Bergman, *J. Phys. C: Solid St. Phys.* **14**:909 (1981); (d) B. D. Hughes and M. F. Shlesinger, *J. Math. Phys.*, **23**:1688 (1982).

2. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd edition (Wiley, New York, 1971).
3. P. Lévy, *Théorie de l'addition des variables aléatoires* (Gauthier-Villars, Paris, 1937).
4. J. E. Gillis and G. H. Weiss, *J. Math. Phys.* **11**:1307 (1970).
5. E. C. Titchmarch, *The Theory of the Riemann Zeta Function* (Oxford, Clarendon Press, 1951).
6. G. Pólya, *Math. Ann.* **84**:149 (1921).
7. K. Lindenberg, V. Seshadri, K. E. Shuler, and G. H. Weiss, *J. Stat. Phys.* **23**:11 (1980).
8. E. W. Montroll, *Proc. Symp. Appl. Math.* **16**:193 (1964).
9. B. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (W. H. Freeman, San Francisco; 1977); and *The Fractal Geometry of Nature* (W. H. Freeman, San Francisco, 1982).
10. B. D. Hughes, M. F. Shlesinger, and E. W. Montroll, *Proc. Nat. Acad. Sci.* (*USA*), **78**:3287 (1981).
11. B. D. Hughes, E. W. Montroll, and M. F. Shlesinger, *J. Stat. Phys.* **28**:111 (1982).
12. M. F. Shlesinger and B. D. Hughes, *Physica* **109A**:597 (1981).
13. W. Hurewicz and H. Wallman, *Dimension Theory* (Princeton University Press, Princeton, New Jersey, 1948).
14. K. Menger, *Amer. Math. Monthly* **50**:2 (1943).
15. H. P. McKean, *Ann. Math.* **61**:564 (1955).
16. R. M. Blumenthal and R. K. Getoor, *Trans. Amer. Math. Soc.* **95**:263 (1960).
17. P. Dienes, *The Taylor Series* (Oxford University Press, Oxford, 1931).
18. J. P. Kahane, *Bull. Amer. Math. Soc.* **70**:193 (1964).
19. E. Fabry, *Annales Sc. de l'École Normale* (3) **13**:107 (1896).
20. G. H. Hardy, *Trans. Amer. Math. Soc.* **17**:301 (1916).
21. E. Neuenschwander, *Math. Intelligencer* **1**:40 (1978); S. L. Segal, *ibid.* **1**:81 (1978).
22. H. M. Edwards, *Riemann's Zeta Function* (Academic Press, New York, 1974).
23. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd edition (Cambridge University Press, 1944).
24. V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, *J. Stat. Phys.* **9**:45 (1973).
25. T. Carleman, *Acta Math.* **59**:63 (1932); E. W. Montroll and R. H. G. Helleman, *AIP Conf. Proc.* **27**:75 (1976); E. W. Montroll, *AIP Conf. Proc.* **46**:337 (1978).
26. T. Nishigori, *J. Math. Phys.* **22**:2903 (1981).